The Fundamental Review of the Trading Book: from VaR to ES

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Table of contents

1 Introduction

2 Value at Risk and Expected Shortfall

3 Sensitivity Analysis

4 Back-Testing Procedures

The Context

After the financial crisis in 2008-2010, the *Basel Committee* tried to sort the unsolved issues in the Basel II. One of the main changes concerned the general market risk requirement, i.e. how we have to measure the unexpected loss of our portfolio.
During the past decade, *Value-at-Risk* (commonly known as **VaR**) has become one of the most popular risk measurement techniques in finance.

The Basel III Committee establishes that in risk measurement **VaR** has to be replaced by the *Expected Shortfall* (**ES**). VaR and ES are used to determine the *capital charge* of a bank, that is the amount of money which the bank must save to cover unexpected losses.

The Context

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Prices, Returns and Portfolio

Generally, we consider a time horizon $t \in [0, M]$ and a portfolio $w = (w_1, \dots, w_N)$ made of N assets, the weights w_i stand for how much money we invest in the asset i. Denoting by $S_{t,i}$ the market price of the asset i at time t the portfolio value at time t is

$$V_t = \sum_i w_i S_{t,i}$$

and the portfolio *Profit-Loss* in time $[0, M]$ is

$$PL = V_M - V_0 = \sum_i w_i (S_{t,i} - S_{0,i}).$$

If we denote by

$$R_{t,i} = \log(S_{t,i}/S_{t-1,i})$$

we can also rewrite the portfolio Profit-Loss as follows

$$PL_t = V_0 \sum_i \theta_i R_{t,i}, \qquad \theta_i = \frac{w_i S_{0,i}}{V_0}.$$

Risk Measures

The **Value at Risk** of order α of the portfolio w, for the time horizon M, is defined as $VaR_\alpha(PL)$ where

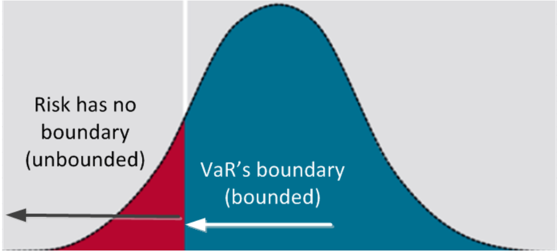
$$VaR_\alpha(PL) = -q_\alpha(PL) \qquad PL \text{ is a random variable}$$

where $q_\alpha(PL)$ is the quantile of order α of the probability function $F(x)$ of PL.

The **Expected Shortfall** of order α of the portfolio w, for the time horizon M, is defined as $ES_\alpha(PL)$ where

$$ES_\alpha(PL) = E[PL|PL \leq VaR_\alpha] = \frac{1}{\alpha}\int_0^\alpha VaR_u(PL)du$$

Risk Measures



Remark (Square Root Rule)

We point out that the time window refers to daily data, then the VaR is daily. If we want to compute VaR_T for a different time horizon T we use the approximation $VaR_T = VaR \cdot \sqrt{T}$.
Such approximation holds exactly as equality when the price process of our Portfolio is driven by a Geometric Brownian Motion such as in the Black-Scholes dynamic.

The Approaches

VaR and ES computation were performed by using four different approaches. We now list the different techniques.

- Parametric models:
  - Δ-Normal approach;
  - Exponential Weighted Moving Average (EWMA) risk metrics,
- Non-parametric models:
  - Historical Simulation;
  - Weighted Historical Simulation.

Δ-Normal approach

Within the Δ-Normal Approach we assume that the vector of the returns R has a multivariate normal distribution, then the vector PL is normally distributed with mean zero and variance

$$\sigma_{PL}^2 = w^T \cdot \Sigma \cdot w$$

where $\Sigma_{ij} = Cov(R_i, R_j)$. In this model it is simple to compute the VaR and the ES of our portfolio

$$\begin{aligned} VaR &= -\sigma_{PL} q_\alpha(Z) & \quad Z \sim \mathcal{N}(0;1); \\ ES &= -\sigma_{PL} \frac{\varphi(q_\alpha)}{\alpha} & \quad \varphi \text{ gaussian density.} \end{aligned}$$

EWMA Risk Metrics

The EWMA approach makes use of the same formulas of the Δ-Normal Approach for the VaR and ES.
It requires to build a weighted Covariance Matrix $\widetilde{\Sigma} = (\tilde{\sigma}_{ij})$ of the returns according to the following definition

$$\tilde{\sigma}_{ij} = \frac{1-\lambda}{1-\lambda^M} \sum_{m=1}^{M} \lambda^{m-1} R_{M-m,i} R_{M-m,j}$$

and replace Σ with $\widetilde{\Sigma}$.

Remark

The EWMA model is similar to GARCH: we can estimate the variance by using a regressive method

$$\sigma_{k+1}^2 = \lambda\sigma_k^2 + (1-\lambda)R_k^2, \qquad \sigma_0 \text{ fixed}$$

where R_k denotes the Profit and Loss of our Portfolio at time k.

Historical Simulation

Historical simulation is a widely used method because

- we do not have to specify any probability model;
- we do not have to estimate any parameter.

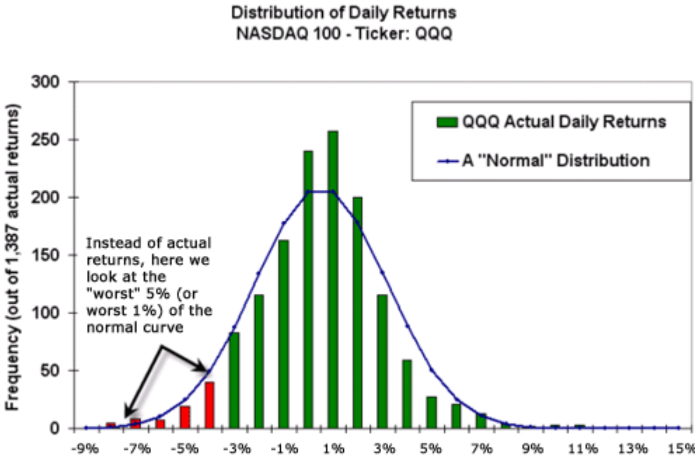
We create the order statistics of PL (from the lowest to the highest) $PLs = (PL_{(1)}, \dots, PL_{(N)})$ and we assume that the theoretical distribution is exactly the empirical one.

Under this Hypothesis it is *very simple* to compute the VaR_α. Denoting by $\beta = [\alpha \cdot N]$ (the integer part of $\alpha \cdot N$), we have

$$VaR_\alpha = PLs_\beta$$

Remark

*The Historical Simulation Method has a **high variance estimator**.*


Distribution of Daily Returns
NASDAQ 100 - Ticker: QQQ
Frequency (out of 1,387 actual returns)
300
250
200
150
100
50
0
-9%
-7%
-5%
-3%
-1%
1%
3%
5%
7%
9%
11%
13%
15%
QQQ Actual Daily Returns
A "Normal" Distribution
Instead of actual returns, here we look at the "worst" 5% (or worst 1%) of the normal curve

Weighted Historical Simulation

We fix some decreasing weights p_m, with $1 < m < N$, meaning that $p_m \in (0,1)$, $\sum_m p_m = 1$ and $p_{m+1} < p_m$. For example

$$p_m = \frac{1-\lambda}{1-\lambda^M}\lambda^{m-1}, \qquad \lambda \in (0,1)$$

and we sort the weight p_m in accordance with the new position of the element PL_m in PLs, we create a vector $p_{sort} = (p_{(1)}, \ldots, p_{(N)})$ and finally the vector $PLw = (p_{(1)} \cdot PL_{(1)}, ..., p_{(N)} \cdot PL_{(N)})$.
For a fixed α we determine m^* as follows

$$\sum_{i=1}^{m^*} PLw_i \leq \alpha < \sum_{i=1}^{m^*+1} PLw_i$$

we set $VaR_\alpha = PLw_{m^*}$.

Our Portfolio

We created a panel including 38 assets, of different nature and came from different geographical areas.

We downloaded the time series from *Yahoo!Finance*. The observed window data starts from 6/9/2014 up to 6/9/2016.

Here the panel composition:

- index: NYA, DAX, IPSA, etc...
- equities: Apple, Ford, Bank of America, ENI, Telecom, etc...

We construct a portfolio $w = (w_1, \ldots, w_{38})$ by picking $w_i = \frac{1}{38}$.

Cleaning and Matching of the data

When we compose a diversified portfolio including for instance index, stocks, bonds, equities, founds and we merge the data in order to made up the matrix of the prices, we can find missing values (*NA*). For example in our case we have 624 NA over 20710 values.

We adopt the following criterium

- As concern the data from 6/9/2014 to 6/8/2016 we split the procedure into two cases
  - If there are more than 15% of NA we erase the row;
  - If there are less then 15% of NA we interpolate the missing data $S_{t,i}$

$$S_{t,i} = \frac{S_{t-1,i} + S_{t+1,i}}{2}(1 + \sigma_i \mathcal{N}(0,1))$$

  where σ_i is the variance of the returns of the asset i.

- As concerns the period from 6/8/2016 to 6/9/2016 we always interpolate the not available data.

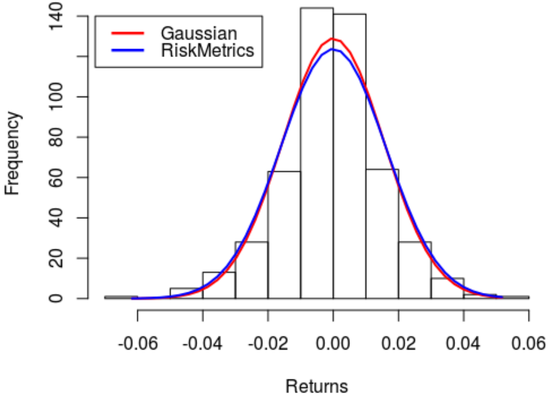
Portfolio Risk Computation

We create the matrix of the log-return R and we compute VaR and ES by using the previous 4 methods with $\alpha = 0.01$

	Δ-Normal	EMWA ($\lambda = 0.99$)	EMWA ($\lambda = 0.94$)
VaR	3.59%	3.74%	2.47%
	Historical	W-Historical ($\lambda = 0.99$)	W-Historical ($\lambda = 0.94$)
VaR	4.27%	4.52%	3.19%

	Δ-Normal	EMWA ($\lambda = 0.99$)	EMWA ($\lambda = 0.94$)
ES	4.12%	4.29%	2.83%
	Historical	W-Historical ($\lambda = 0.99$)	W-Historical ($\lambda = 0.94$)
ES	4.81%	5.10%	3.93%

Histogram of returns



$$\sigma_G = 1,5\%$$

$$\sigma_{RM} = 1,6\%$$

We performed the Jarque-Bera test which said that the *PL* distribution is Normal.

Historical Vs W-Historical

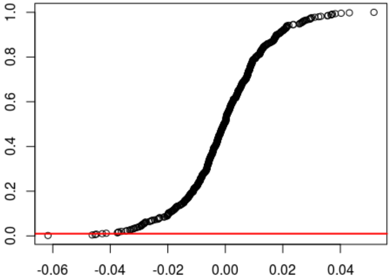


Figure: Empirical cumulative function (Historical)

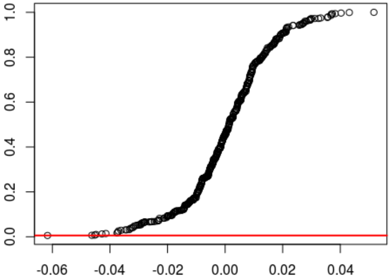


Figure: Empirical cumulative function (W-Historical)

Perturbation of Weights in Portfolio

This analysis help us to understand which instruments are more/less risky. We use the following notations

- $w = (w_1, w_2, \dots, w_N)$, and we start by picking uniform weights:

$$w_1 = w_2 = \dots = w_N,$$

- i - index of instrument,
- r - factor change,
- $w' = \frac{(w_1,\dots,w_i r,\dots,w_N)}{\|(w_1,\dots,w_i r,\dots,w_N)\|}$,

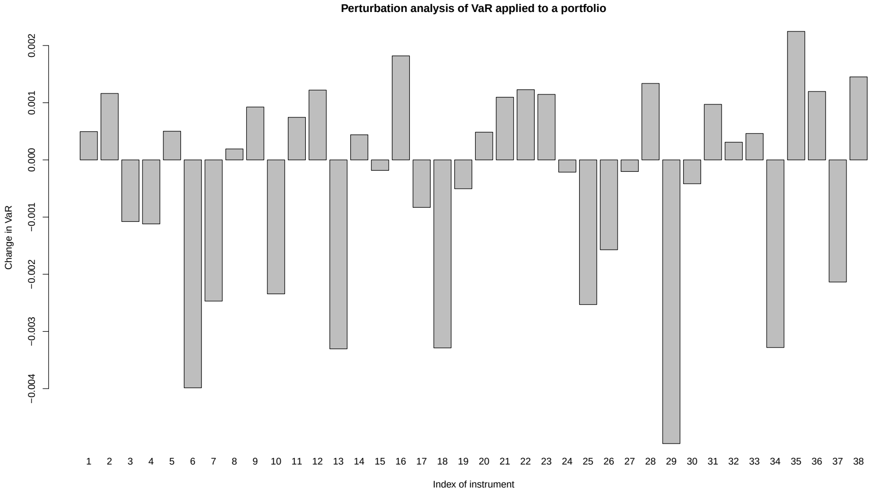
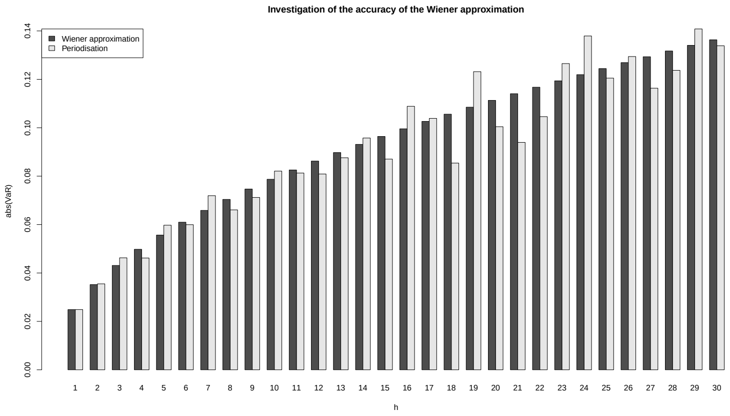


Figure: Perturbation Analysis

Wiener Process Approximation in the VaR Estimation over Time Horizon h

- $\text{VaR}(\alpha, h)$
- h - time horizon ($h = 1$),
- Wiener process approximation (Brownian motion),
- $h \neq 1 \rightarrow \text{VaR}(\alpha, h) \approx \text{VaR}(\alpha, 1)\sqrt{h}$,
- Alternative: divide data into h long intervals and estimate $\text{VaR}(\alpha, h)$.



The figure confirm the small deviations of empiric data from the *square root rule*, then we can conclude that the price process of our Portfolio can be modeled by a *Geometric Brownian Motion.*

Backtesting of VaR

In order to use a quantitative method for the risk measurement, the banks must satisfy some backtesting requirements, that means the model must show to be predictive when it is used in the day-by-day process.
Consider the event that the loss on a portfolio at time $t+1$ exceeds its reported VaR, $VaR_t(\alpha)$

$$I_{t+1}(\alpha) = \begin{cases} 1 & \text{if } R_{t+1} \leq VaR_t(\alpha) \\ 0 & \text{if } R_{t+1} > VaR_t(\alpha) \end{cases} \qquad \text{e.g. } (0,0,1,0,\ldots,0,1)$$

For being an accurate risk measure Christofferson(1998) stated two properties for the hit sequence I_t:

- Unconditional Coverage Property
- Independence Property

This results in the assumption $\quad I_t \overset{i.i.d}{\sim} B(\alpha)$.

Kupiec Test

The Kupiec test focuses on the unconditional property.
It concerns whether VaR is violated more than $\alpha \times 100\%$ of the time.
Using a sample of T observations we define

$$\hat{\alpha} = \frac{1}{T} \sum_{t=1}^{T} I_t(\alpha)$$

Null Hypothesis

$$H_0 \quad : \quad \hat{\alpha} = \alpha$$

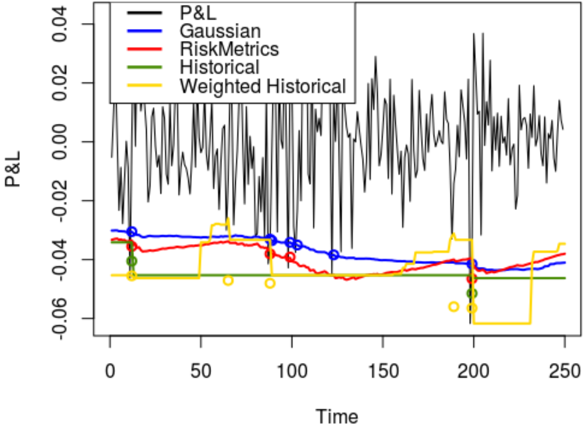


Figure: **Value at Risk** Number of Failures: 7, 4, 2, 5.

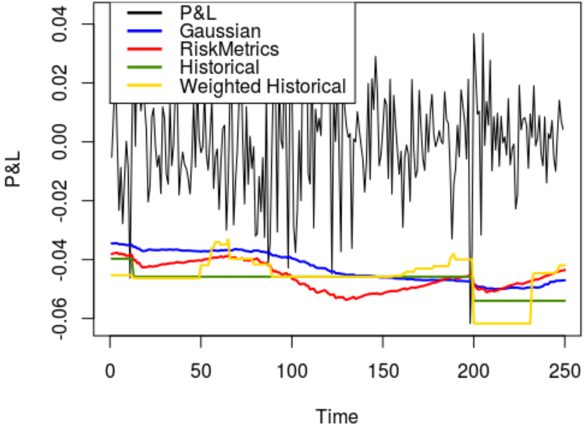


Figure: **Expected Shortfall**

Towards the ES Back-Test

The discovery in 2011 that the ES is not *elicitable* as emphasized by *Gneiting "Marking and Evaluating Point Forecasts"*, diffused the erroneous belief that it could not be Back-Tested.
It is a fact that the absence of a convincing back-test has long been the last obstacle for ES on its way to Basel. The migration from VaR to ES was criticized.

Only in 2014 Acerbi and Szakely found three different tests to beck-test the ES. We refer to their paper *"Backtesting Expected Shortfall"* for further details.

Towards the ES Back-Test

Acerbi-Szakely Statistics

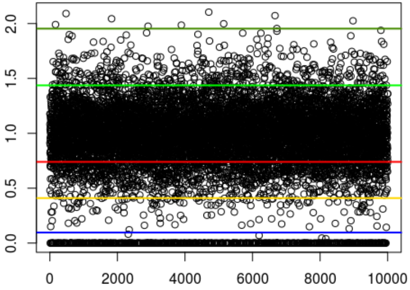
$$Z = \frac{1}{N_{\mathbf{1}}} \cdot \sum_{t=1}^{M} \frac{\mathbf{1}_t R_t}{ES_t} + 1$$

where

- $\mathbf{1}_t$ is the indicator function of the event $\{R_t < VaR_t\}$
- $N_{\mathbf{1}} = \sum_t \mathbf{1}_t$

The estimator is itself a Random Variable and its distribution is unknown. We perform the test by using a Monte Carlo approximation of the distribution under the H_0 hypothesis.

Towards the ES Back-Test



The Back-Testing shows good figures as the observed is strictly below of the blue rejection level, but the key point is that we achieved an effective ES backtestability.