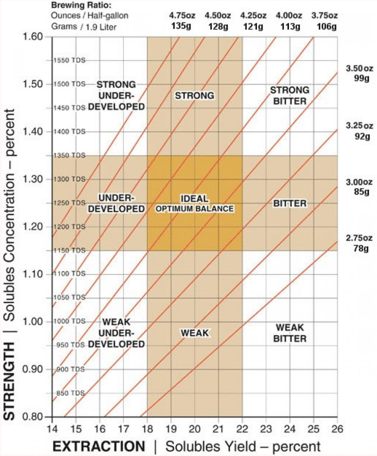
Brewing Filter Coffee:
Mathematical Model of Coffee Extraction

Modelling Camp, ICMS
March 24, 2016


PHILIPS

The Problem


Brewing Ratio:
Ounces / Half-gallon
Grams / 1.9 Liter
4.75oz 135g
4.50oz 128g
4.25oz 121g
4.00oz 113g
3.75oz 106g
3.50oz 99g
3.25oz 92g
3.00oz 85g
2.75oz 78g
STRONG UNDER-DEVELOPED
STRONG
STRONG BITTER
UNDER-DEVELOPED
IDEAL
OPTIMUM BALANCE
BITTER
WEAK UNDER-DEVELOPED
WEAK
WEAK BITTER
1550 TDS
1500 TDS
1450 TDS
1400 TDS
1350 TDS
1300 TDS
1250 TDS
1200 TDS
1150 TDS
1100 TDS
1050 TDS
1000 TDS
950 TDS
900 TDS
850 TDS
1.60
1.50
1.40
1.30
1.20
1.10
1.00
0.90
0.80
STRENGTH | Solubles Concentration – percent
14 15 16 17 18 19 20 21 22 23 24 25 26
EXTRACTION | Solubles Yield – percent

Outline

- Examining the concentration of granules vs coffee in solution.
- Model the flow through the coffee-bed.
- Simplify model of extraction with advection in the filter.
- Exciting Results!

Outline

- Examining the concentration of granules vs coffee in solution.
- Model the flow through the coffee-bed.
- Simplify model of extraction with advection in the filter.
- Exciting Results!

Outline

- Examining the concentration of granules vs coffee in solution.
- Model the flow through the coffee-bed.
- Simplify model of extraction with advection in the filter.
- Exciting Results!

Outline

- Examining the concentration of granules vs coffee in solution.
- Model the flow through the coffee-bed.
- Simplify model of extraction with advection in the filter.
- Exciting Results!

Outline

- Examining the concentration of granules vs coffee in solution.
- Model the flow through the coffee-bed.
- Simplify model of extraction with advection in the filter.
- Exciting Results!

Variables

Basic Variables:

$$C_c := \frac{m_c}{V\theta}, \quad C_g := \frac{m_g}{V(1-\theta)},$$

where: C_c represents the concentration of the coffee in water
C_g the concentration of the coffee granules
θ is the porosity of the coffee

Basic Equations

Equation of transport of coffee for constant density of water at a certain temperature:

$$\frac{dC_c}{dt} = \alpha(1-\theta)(C_g - G\lambda)(S - C_c) - (v_w \cdot \nabla C_c)$$

Conservation of coffee granules

$$\frac{d}{dt}\left(\theta C_c + (1-\theta)C_g\right) = 0$$

$$\implies \theta C_c + (1-\theta)C_g = (1-\theta)G$$

G is the starting concentration of granules, and S is the maximum concentration of dissolved coffee, α is the extraction rate.

Basic Equations

Equation of transport of coffee for constant density of water at a certain temperature:

$$\frac{dC_c}{dt} = \alpha(1-\theta)(C_g - G\lambda)(S - C_c) - (v_w \cdot \nabla C_c)$$

Conservation of coffee granules

$$\frac{d}{dt}\left(\theta C_c + (1-\theta)C_g\right) = 0$$

$$\implies \theta C_c + (1-\theta)C_g = (1-\theta)G$$

G is the starting concentration of granules, and S is the maximum concentration of dissolved coffee, α is the extraction rate.

Basic Equations

Equation describing the coffee concentration within the granules:

$$\frac{dC_g}{dt} = -\theta\alpha(C_g - G\lambda)(S - C_c)$$

Basic Equations

Equation describing the coffee concentration within the granules:

$$\frac{dC_g}{dt} = -\theta\alpha(C_g - G\lambda)(S - C_c)$$

Dimensionless System

Dimensionless system without advection:

$$\frac{d\widetilde{C}_c}{dt} = B(1-\theta)G(\widetilde{C}_g - \lambda)(1 - \widetilde{C}_c)$$
$$\frac{d\widetilde{C}_g}{dt} = -\theta(\widetilde{C}_g - \lambda)(1 - \widetilde{C}_c),$$

where $\widetilde{C}_c := \frac{C_c}{S}$, $\widetilde{C}_g := \frac{C_g}{G}$, $\widetilde{t} = t/T$ and $B = G/S$

Dimensionless System

Dimensionless system without advection:

$$\frac{d\widetilde{C}_c}{dt} = B(1-\theta)G(\widetilde{C}_g-\lambda)(1-\widetilde{C}_c)$$
$$\frac{d\widetilde{C}_g}{dt} = -\theta(\widetilde{C}_g-\lambda)(1-\widetilde{C}_c),$$

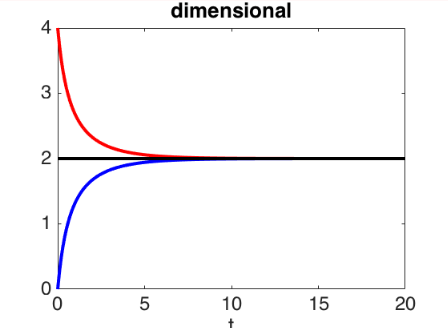
where $\widetilde{C}_c := \frac{C_c}{S}$, $\widetilde{C}_g := \frac{C_g}{G}$, $\widetilde{t} = t/T$ and $B = G/S$

Results for the concentrations

$$\widetilde{C}_c = \frac{B(1-\theta)}{\theta}(1-\widetilde{C}_g)$$

$$\widetilde{C}_g = \frac{\lambda\theta + (1-\lambda)(\theta - B(1-\theta))e^{-\widetilde{j}t}}{\theta + (1-\lambda)(1-\theta)Be^{-\widetilde{j}t}},$$

where $\widetilde{j} = \theta - (1-\lambda)B(1-\theta)$



Flow Through the Coffee-Bed

Darcy's law describes the flow of water through the coffee (porous medium)

$$q = -\frac{k}{\mu}\nabla P$$

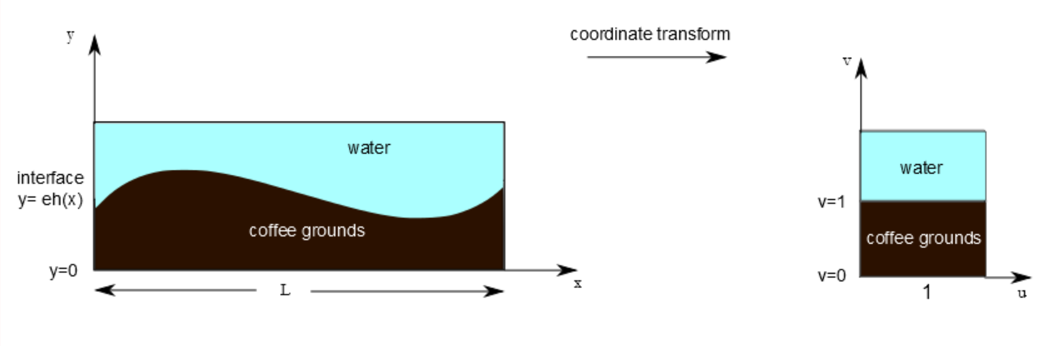


Figure: $x = Lu\ ,\ y = h(u)v$.

Pressure-Velocity

$$\text{Pressure: } P = \rho_w g y \left(\frac{H}{h(x)} - 1 \right) + P_0$$

$$\text{Velocity: } v_y = \frac{-\kappa}{\theta\mu} \rho_w g \left(\frac{H}{h(x)} - 1 \right)$$

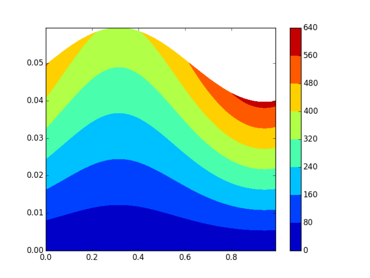
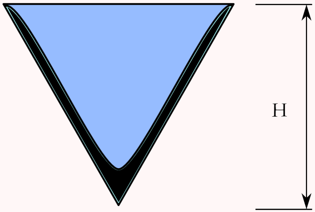
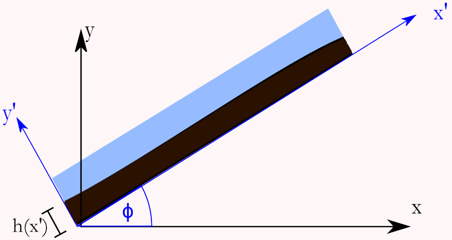


Figure: Pressure Distribution

Rotating the Problem





Pressure: $P = \rho_w g h^{-1} y' \left(H - x' \sin(\phi) - h(x') \cos(\phi) \right)$

Rotating the Problem

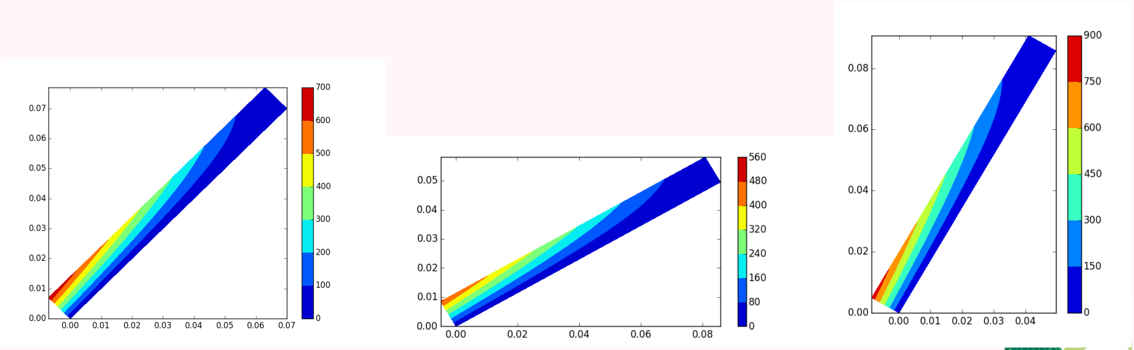


Figure: Pressure distribution at inclination angle 45, 30, 60 respectively

Mean-field Approximation

Average over coffee bed height:

$$\widetilde{C}_c = \frac{1}{h}\int_0^h C_c \, dz$$
$$\widetilde{C}_g = \frac{1}{h}\int_0^h C_g \, dz$$

$$\frac{1}{h}\int_0^h (\nabla \cdot v_w C_c)\, dz = v_w \left(C_c(h) - C_c(0)\right)$$
$$= -v_w \widetilde{C}_c$$

Mean-field approximation:

$$\frac{1}{h}\int_0^h f(C_c, C_g) dz \approx f(\widetilde{C}_c, \widetilde{C}_v)$$

Average $\widetilde{C}_c$ and $\widetilde{C}_g$

Average over volume using mean-field argument:

$$\int_0^h \left[\frac{\partial C_c}{\partial t} + \nabla \cdot (C_c v_w)\right] dz = \int_0^h \alpha(1-\theta)(C_g - G\lambda)(S - C_c) dz$$

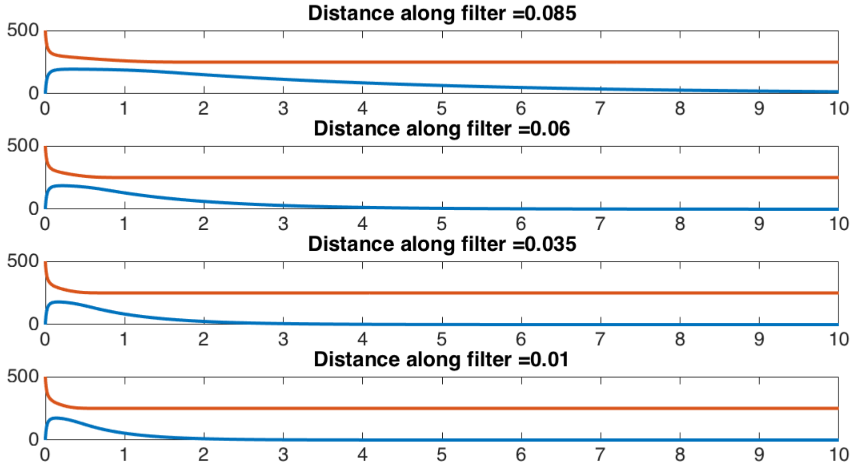
$$\frac{\partial \hat{C}_c}{\partial t} - v_w \hat{C}_c = \alpha(1-\theta)(\hat{C}_g - G\lambda)(S - \hat{C}_c)$$

$$\int_0^h \frac{\partial C_g}{\partial t} dz = \int_0^h -\alpha\theta(C_g - G\lambda)(S - C_c) dz$$

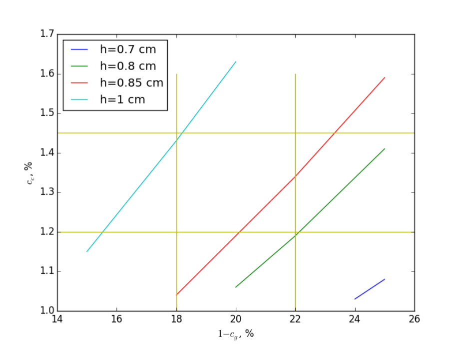
$$\frac{\partial \hat{C}_g}{\partial t} = -\alpha\theta(\hat{C}_g - G\lambda)(S - \hat{C}_c)$$

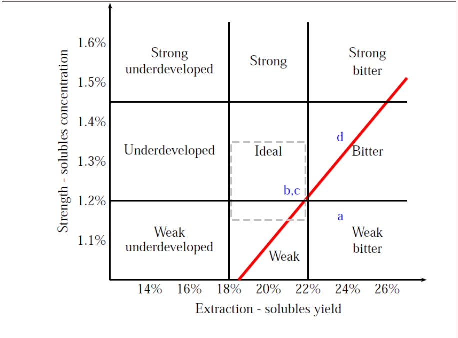
Illustration of the solution with advection

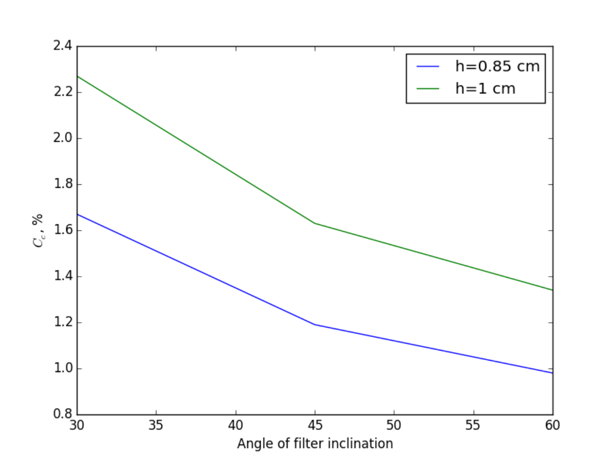
C_c-blue curve, C_g red curve



Brewing Contral Chart Comparison







h=0.85 cm
h=1 cm
C_c, %
Angle of filter inclination
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
30
35
40
45
50
55
60

Conclusions and future research

- We developed a basic model, which for a given geometry of the coffee bed predicts quality of the coffee
- More coffee is extracted at the top of the filter rather than at the bottom due to the lower pressure and lower velocity
- An decrease in the angle of inclination of the filter leads to an increase in the concentration of coffee in the solution
- Our model predicts the height of the coffee bed along the filter should be in the range $0.8 < h < 1$ cm
- Straightforward extensions: 3D axisymmetric model, variable h
- Further improvements: consider the process of a coffee bed deformation and chemical impact

Conclusions and future research

- We developed a basic model, which for a given geometry of the coffee bed predicts quality of the coffee
- More coffee is extracted at the top of the filter rather than at the bottom due to the lower pressure and lower velocity
- An decrease in the angle of inclination of the filter leads to an increase in the concentration of coffee in the solution
- Our model predicts the height of the coffee bed along the filter should be in the range $0.8 < h < 1$ cm
- Straightforward extensions: 3D axisymmetric model, variable h
- Further improvements: consider the process of a coffee bed deformation and chemical impact

Conclusions and future research

- We developed a basic model, which for a given geometry of the coffee bed predicts quality of the coffee
- More coffee is extracted at the top of the filter rather than at the bottom due to the lower pressure and lower velocity
- An decrease in the angle of inclination of the filter leads to an increase in the concentration of coffee in the solution
- Our model predicts the height of the coffee bed along the filter should be in the range $0.8 < h < 1$ cm
- Straightforward extensions: 3D axisymmetric model, variable h
- Further improvements: consider the process of a coffee bed deformation and chemical impact

Conclusions and future research

- We developed a basic model, which for a given geometry of the coffee bed predicts quality of the coffee
- More coffee is extracted at the top of the filter rather than at the bottom due to the lower pressure and lower velocity
- An decrease in the angle of inclination of the filter leads to an increase in the concentration of coffee in the solution
- Our model predicts the height of the coffee bed along the filter should be in the range $0.8 < h < 1$ cm
- Straightforward extensions: 3D axisymmetric model, variable h
- Further improvements: consider the process of a coffee bed deformation and chemical impact

Conclusions and future research

- We developed a basic model, which for a given geometry of the coffee bed predicts quality of the coffee
- More coffee is extracted at the top of the filter rather than at the bottom due to the lower pressure and lower velocity
- An decrease in the angle of inclination of the filter leads to an increase in the concentration of coffee in the solution
- Our model predicts the height of the coffee bed along the filter should be in the range $0.8 < h < 1$ cm
- Straightforward extensions: 3D axisymmetric model, variable h
- Further improvements: consider the process of a coffee bed deformation and chemical impact

Conclusions and future research

- We developed a basic model, which for a given geometry of the coffee bed predicts quality of the coffee
- More coffee is extracted at the top of the filter rather than at the bottom due to the lower pressure and lower velocity
- An decrease in the angle of inclination of the filter leads to an increase in the concentration of coffee in the solution
- Our model predicts the height of the coffee bed along the filter should be in the range $0.8 < h < 1$ cm
- Straightforward extensions: 3D axisymmetric model, variable h
- Further improvements: consider the process of a coffee bed deformation and chemical impact

Conclusions and future research

- We developed a basic model, which for a given geometry of the coffee bed predicts quality of the coffee
- More coffee is extracted at the top of the filter rather than at the bottom due to the lower pressure and lower velocity
- An decrease in the angle of inclination of the filter leads to an increase in the concentration of coffee in the solution
- Our model predicts the height of the coffee bed along the filter should be in the range $0.8 < h < 1$ cm
- Straightforward extensions: 3D axisymmetric model, variable h
- Further improvements: consider the process of a coffee bed deformation and chemical impact