Bottle Testing

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- Future Work

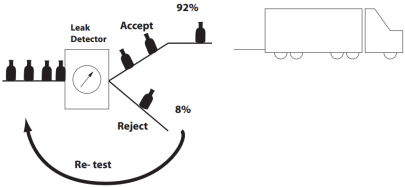
The Problem

The current leak test is noisy:

- 8% fail the original test

BUT

- nearly all of these fails are judged to be fine after the secondary test
- only about 0.4% fail the secondary test.



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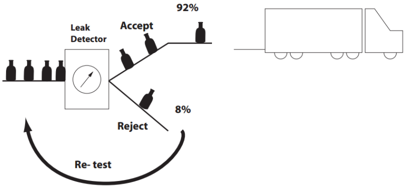

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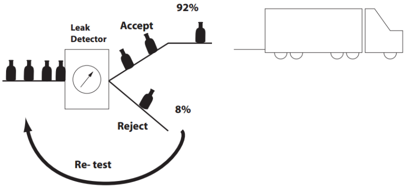
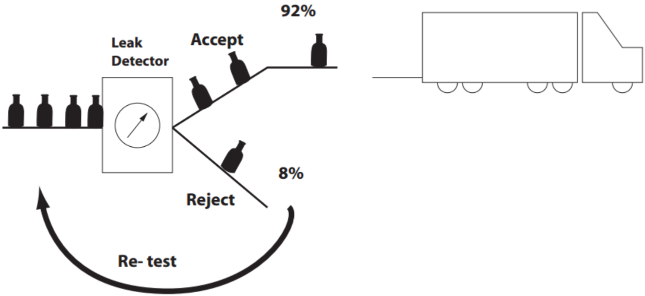


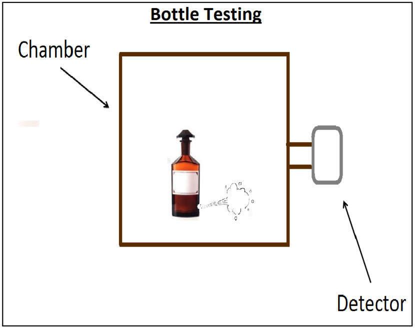
* Thus, too many good bottles are being initially rejected!
* The secondary test is expensive.

The Problem

QUESTION: How do we reduce the number of good bottles rejected, without more bad bottles being accepted?

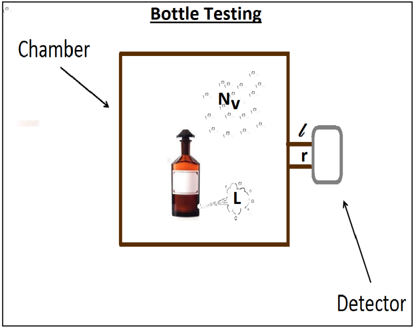


Company's Testing Procedure



- Each bottle is filled with He before entering the chamber.
- The mass spectrometer detects the He that leaks from the bottle.
- If He concentration goes above a certain threshold the bottle is rejected, it is tested again.

Formulation of the Model



Change in amount of He
in the chamber over time
=
Leakage - Amount of He that leaves the chamber and detected

$$\frac{dN_v}{dt} = L - aN_v$$

Formulation of the Model

$$\frac{dN_v}{dt} = \mathsf{L} - aN_v$$

with

$$\mathsf{L} = \frac{PA}{\sqrt{2\pi k_B m T}}$$

where

$$P = \textit{pressure of bottle}$$
$$A = \textit{area of the hole}$$
$$m = \textit{mass of a single He molecule}$$
$$k_B = \textit{Boltzmann constant}$$
$$T = \textit{temperature}$$

Formulation of the Model

$$\boxed{\frac{dN_v}{dt} = L - \mathsf{a} N_v}$$

with

$$\mathsf{a} = \frac{\textit{Volume flow rate}}{\textit{Volume of tube}} = \frac{r^2 k_B T N_v}{8\eta l^2 V_v}$$

where

$$\begin{aligned} r &= \textit{radius of the tube} \\ k_B &= \textit{Boltzmann constant} \\ T &= \textit{temperature} \\ \eta &= \textit{viscosity} \\ l &= \textit{length of tube} \\ V_v &= \textit{volume of the chamber} \end{aligned}$$

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Formulation and Solution of the Deterministic Model

$$\boxed{\frac{dN_v}{dt} = L - cN_v^2}$$

with

$$L = \frac{PA}{\sqrt{2\pi k_B m T}} = \textit{constant}, \quad c = \frac{r^2 k_b T}{8nl^2 V_v} = \textit{constant}$$

The solution is

$$N_v = \sqrt{\frac{L}{c}}\tanh(\sqrt{Lc}t)$$

Statistical Model

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- Measurements at t_i where $i = 1, ..., n$.

$$D(t_i) = cN(t_i) + \varepsilon_i \quad \textit{where} \;\; \varepsilon_i \sim \mathcal{N}(0, \sigma^2) \quad \textit{i.i.d}$$

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For large n, $\hat{L}$ is asymptotically normal:

$$\hat{L} \sim \mathcal{N}\left(L, \frac{1}{nI(L)}\right), \text{ where } I = \text{Fisher's Information.}$$

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Probability of false acceptance?

$$P(\hat{L} < L_1 | L > L_0) = \frac{\int_{L_0}^{\infty} P(\hat{L} < L_1 | L = y)\mu(dy)}{P(L > L_0)}$$

Stochastic Model

Stochastic Model

Since the deterministic model does not reflect the "actual truth", the stochastic can give more insights for the problem. We consider

the following mean reverting process:

$$\boxed{dZ(t) = c dN(L;t) + (cN^2(L;t) - Z(t))dt + bZ(t)dW_t}$$

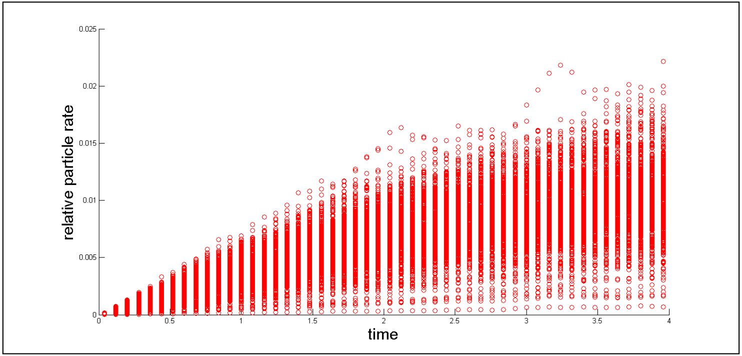
assuming that $L \sim \mathcal{N}(1, 0.1)$.

Limitation of Data

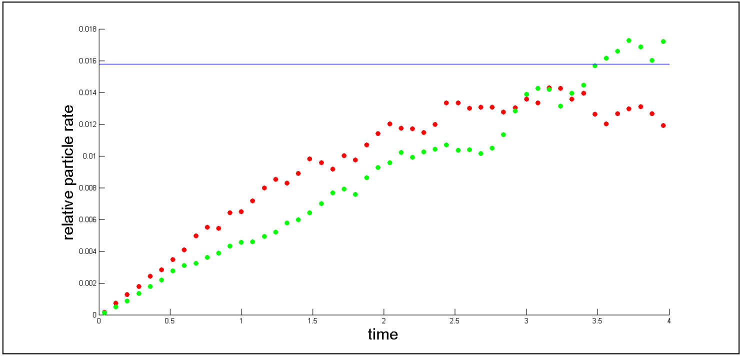
Data provided by the company:

$$P(r_1) = 8\% \;\; \textit{and} \;\; P(r_2) \approx 0.5\%$$

Group's Stochastic Model Outcome



Company's Testing Method



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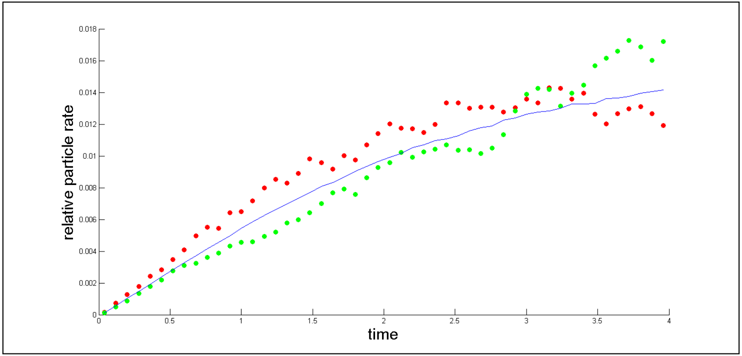

Results

METHOD:	Company's	Group's
BAD BOTTLES ACCEPTED	3	1
GOOD BOTTLES REJECTED	163	84

* Results out of 2000 bottles tested.

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In other words, Group's method:

- cut out the probability of accepting a bad bottle to $1/3$; and,
- cut out the probability of rejecting a good bottle to $1/2$.

Conclusion/Future Work

- Group's model cannot eliminate the probability that the good (bad) bottle is rejected (accepted).

BUT

- Group's model is **MORE ACCURATE** than the existing model used by the company.
- Provided the data one can use both statistical and stochastic models as a base in order to produce more accurate results for the company's benefit.

Some References

- Y.Deng et al. (2013) "Residual Useful Life Estimation Based on a Time-Dependent Ornstein Uhlenbeck Process"
- C. Vallance (lecture Notes) "Properties of Gases".

Thank you!